V. "Second Memoir on the Curves which satisfy given conditions; the Principle of Correspondence." By Professor CAYLEY, F.R.S. Received April 18, 1867.

(Abstract.)

In the present memoir I reproduce with additional developments the theory established in my paper "On the Correspondence of two points on a Curve" (London Math. Society, No. VII., April 1866); and I endeavour to apply it to the determination of the number of the conics which satisfy given conditions; these are conditions of contact with a given curve, or they may include arbitrary conditions Z, 2Z, &c. If, for a moment, we consider the more general question where the Principle is to be applied to finding the number of the curves $C^r$ of the order $r$, which satisfy given conditions of contact with a given curve, there are here two kinds of special solutions; viz., we may have proper curves $C^r$ touching (specially) the given curve at a cusp or cusps thereof, and we may have improper curves, that is, curves which break up into two or more curves of inferior orders. In the case where the curves $C^r$ are lines, there is only the first kind of special solution, where the sought for lines touch at a cusp or cusps. But in the case to which the memoir chiefly relates, where the curves $C^r$ are conics, we have the two kinds of special solutions, viz., proper conics touching at a cusp or cusps, and conics which are line-pairs or point-pairs. In the application of the Principle to determining the number of the conics which satisfy any given conditions, I introduce into the equation a term called the "Supplement" (denoted by the abbreviation "Supp."), to include the special solutions of both kinds. The expression of the Supplement should in every case be furnished by the theory; and this being known, we should then have an equation leading to the number of the conics which properly satisfy the prescribed conditions; but in thus finding the expression of the Supplements, there are difficulties which I am unable to overcome; and I have contented myself with the reverse course, viz., knowing in each case the number of the proper solutions, I use these results to determine *à posteriori* in each case the expression of the Supplement; the expression so obtained can in some cases be accounted for readily enough, and the knowledge of the whole series of them will be a convenient basis for ulterior investigations.

*May* 9, 1867.

Lieut.-General SABINE, President, in the Chair.

Pursuant to notice given at the last Meeting, Mr. Webster proposed, and Mr. Heywood seconded, the Right Hon. Sir William Bovill, Lord Chief Justice of the Common Pleas, for election and immediate ballot.

The ballot having been taken, the Lord Chief Justice Bovill was declared duly elected a Fellow of the Society.

The following communications were read:—

I. "On the Development and Succession of the Teeth in the Marsupialia." By WILLIAM HENRY FLOWER, F.R.S., F.R.C.S., &c., Conservator of the Museum of the Royal College of Surgeons of England.

(Abstract.)

Although the dentition of adult individuals of all the animals which constitute the remarkable Order or, rather, Subclass Marsupialia, has been repeatedly subjected to examination, and described with exhaustive minuteness of detail, it is a singular circumstance that most of those peculiarities in the succession of their teeth which distinguish them from other mammals appear hitherto to have escaped observation. To supply this blank is the object of the present communication. Fortunately the materials at my disposal, although not quite so complete as might be desired, are yet amply sufficient to illustrate the main aspects of the question, and to supply a result as interesting as it was unexpected.

Descriptions are given in the paper, accompanied by drawings, of several stages of the dentition of members of each of the six natural families into which the order is divided.

1. *Macropodidæ.*—The dentition of the Kangaroo (genus *Macropus*), from the completely edentulous fœtus to adult age, is described in detail. Contrary to what has been specially stated with regard to this genus, there are no deciduous or milk-incisors, the teeth of this group which are first formed and calcified in both jaws being those which are retained throughout the life of the animal. The rudimentary canine and first premolar have also no deciduous predecessors. The second tooth of the molar series (a true molar in form) is vertically displaced by a premolar. The four true molars have, as has long been known, no deciduous predecessors. There is thus but one tooth on each side of each jaw in which the phenomenon of diphyodont succession occurs. The period at which this takes place varies in different species of the family. In some forms of *Hypsiprymnus*, the successional premolar is not cut until after the last true molar is in place and use,—this probably having relation to the extraordinary size of the tooth, and the time consequently required for its development. A special characteristic of this family is the tendency to lose the canine and one or both premolars at a comparatively early period of life.

2. *Phalangistidæ.*—Several early stages of the dentition of *Phalangista vulpina* are described and figured. In a young specimen in which no teeth had cut the gum, the crowns of the permanent incisors, canine, and first two molars were found to be calcified, and the germ of the permanent premolar was already formed beneath the milk- or deciduous molar, which, as in *Macropus*, is the only tooth which is shed and replaced by a successor. The change takes place at an earlier period than in the last family.

3. *Peramelidæ.*—No very early stages of *Perameles* were examined; but

adolescent specimens of this genus and of *Chæropus* show that a very minute, compressed, molariform tooth is replaced by the triangular, pointed, third or posterior premolar. No other signs of vertical displacement and succession were observed.

4. *Didelphidæ.*—In the American genus *Didelphys*, the observations are complete from the earliest stage, and show that, as in the Australian *Macropodidæ* and *Phalangistidæ*, none of the teeth of the permanent series have predecessors except the compressed pointed last premolar, which replaces a tooth having the broad multicuspidate crown of a true molar. This change does not occur until the animal approaches the adult age.

5. *Dasyuridæ.*—In a fœtal *Thylacinus*, in which no teeth had cut the gum, the crowns of the permanent incisors, canines, premolars, and anterior true molars were partially calcified, and necessarily much crowded together in the jaw. A very minute rudimentary molar was situated just beneath the alveolar mucous membrane, superficially to the apex of the hindermost premolar, and was evidently its milk-predecessor.

6. *Phascolomyidæ.*—This family is placed last because the observations regarding it are less complete than in the case of any of the others. The youngest Wombat available presented no evidence of succession of any of the teeth; but it is probable that the single premolar is preceded by a milk-molar, at a still earlier period than any examined.

From the foregoing observations it may be concluded with tolerable safety that the animals of the Order Marsupialia present a peculiar condition of dental succession, uniform throughout the order, and distinct from that of all other mammals. This peculiarity may be thus briefly expressed. The teeth of Marsupials do not vertically displace and succeed other teeth, with the exception of a single tooth on each side of each jaw. The tooth in which a vertical succession takes place is always the corresponding or homologous tooth, being the hindermost of the premolar series*, which is preceded by a tooth having the characters, more or less strongly expressed, of a true molar.

It has been usual to divide the class Mammalia, in regard to the mode of formation and succession of their teeth, into two groups—the *Monophyodonts*, or those that generate a single set of teeth, and the *Diphyodonts*, or those that generate two sets of teeth; but even in the most typical diphyodonts the successional process does not extend to the whole of the teeth, always stopping short of those situated most posteriorly in each series. The Marsupials occupy an intermediate position, presenting as it were a rudimentary diphyodont condition, the successional process being confined to a single tooth on each side of each jaw. This position, however, is by no means without analogy among the mammals of the placental series. In the Dugong and the existing Elephants the successional process is limited

* The convenient distinction between false molars or premolars and true molars, is always well marked in the form of the crown, especially in the upper jaw, in the Marsupials.

to the incisor teeth. It is questionable whether the first premolar of those animals of this group which have four premolar teeth, as the Hog, Dog (mandible), &c., ever has a deciduous predecessor, at all events so far advanced as to have reached the calcified stage. But the closest analogy with the marsupial mode of succession is found among the Rodents. Here the incisors appear to have no deciduous predecessors; and in the Beaver, Porcupine, and others, which have but four teeth of the molar series, *i. e.* three true molars and one premolar, the latter is, exactly as in the Marsupials, the only tooth which succeeds a deciduous tooth. The analogy, however, does not hold in those Rodents which have more than one premolar, as the Hare; for in this case each of these teeth has its deciduous predecessor.

In the preceding account I have used the term "permanent" for those teeth which remain in use throughout the animal's life, or, if they fall out (as do the rudimentary canines and the premolars of the *Macropodidæ*), do not give place to successional teeth; and I have therefore assumed that the milk or temporary dentition of the typical diphyodont mammals is represented in the Marsupials only by the deciduous molars. It may be held, on the other hand, that the large majority of the teeth of the Marsupials are the homologues of the milk or first teeth of the diphyodonts, and that it is the permanent or second dentition which is so feebly represented by the four successional premolars. This view is supported by many general analogies in animal organization and development, such as the fact that the permanent state of organs of lower animals often represents the immature or transitional condition of the same parts in beings of higher organization.

Looking only to the period of development of the different teeth in some of the marsupial genera, we might certainly be disposed to place the successional premolar in a series by itself, although, indeed, all its morphological characters point out its congruity with the row of teeth among which it ultimately takes its place, the reverse being the case with its predecessor. It is, however, almost impossible, after examining the teeth of the young Thylacine described and figured in the paper, to resist the conclusion originally suggested. The unbroken series of incisors, canines, premolars, and anterior true molars of nearly the same phase of development, with posterior molars gradually added as age advances, form a striking contrast to the temporary molar, so rudimental in size, and transient in duration. I can scarcely doubt that the true molars of this animal would be identified by every one as homologous with the true molars of the diphyodonts, which are generally regarded as belonging to the permanent series, although they never have deciduous predecessors. Now, if the homology between the true molars of the Thylacine and those of a Dog, for instance, be granted, and if the anterior teeth (incisors, canines, and premolars) of the Thylacine be of the same series as its own true molars, they must also be homologous with the corresponding permanent teeth of the Dog.

It may be objected to this argument, that the true molars of the diphyodonts, not being successional teeth, ought to be regarded as members of the

first or milk- series; but, in truth, the fact that they have themselves no predecessors does not make them serially homologous with the predecessors of the other teeth, while their morphological characters, as well as their habitual persistence throughout life, range them with the second or permanent series.

We have been so long accustomed to look upon the second set of teeth as an after-development or derivative from the first, that it appears almost paradoxical to suggest that the milk- or deciduous teeth may rather be a set superadded to supply the temporary needs of mammals of more complex dental organization. But it should be remembered that, instead of there being any such relation between the permanent and the milk-teeth as that expressed by the terms "progeny" and "parent" (sometimes applied to them), they are both (if all recent researches into their earlier development can be trusted) formed side by side from independent portions of the primitive dental groove, and may rather be compared to twin brothers, one of which, destined for early functional activity, proceeds rapidly in its development, while the other makes little progress until the time approaches when it is called upon to take the place of its more precocious *locum tenens*.

Many facts appear to point to the milk-teeth as being the less constant and important of the two sets developed in diphyodont dentition. Among these the most striking is the frequent occurrence of this set in a rudimentary and functionless or, as it were, partially developed state. The milk-premolars of some Rodents (as the Guinea-pig), shed while the animal is *in utero*, the simple structure and evanescent nature of the milk-teeth of the Bats, Insectivores, and Seals, the diminutive first incisors of the Dugongs and Elephants, all appear to be cases in point. On the other hand, examples of the commencing or sketching out, as it were, of the successors to a well-formed, regular, and functional first set of teeth, are rarely, if ever, met with. Occasional instances of the habitual early decadence, or, perhaps, absence of some of the second or so-called permanent teeth occur in certain animals; but these are rather examples of the disappearance or suppression of organs of which there is no need in the economy, and chiefly occur in isolated and highly modified members of groups in the other members of which the same phenomenon does not occur, as in *Cheiromys* among the Lemurs, *Trichechus* among the Seals, and the recent Elephants (as regards the premolars) among the Proboscideans. They form no parallel to the cases mentioned above of the rudimentary formation of an entire series of teeth of the temporary or milk-set.

To return to the marsupials:—If this view be correct, I should be quite prepared to find, in phases of development earlier than those yet examined, some traces either of the papillary, follicular, or saccular stages of milk-predecessors to other of the teeth besides those determinate four in which, for some reason at present unexplained, they arrive at a more mature growth*. Such proof as this would alone decide the truth of these specu-

* It may be remarked that the milk-tooth, which alone is developed in the Marsu-

lations; and I have not at present either the requisite leisure or materials for following out so delicate an investigation. I trust that the facts already elicited are sufficiently novel and important to justify my bringing them, as they now stand, before the Society.

## II. "On a Property of Curves which fulfil the condition $\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}=0$." By W. J. Macquorn Rankine, C.E., LL.D., F.R.SS.L. & E. Received April 9, 1867.

1. In a paper "On Stream-Lines," published in the Philosophical Magazine for October 1864, I stated, and, in a Supplement to the same paper, published in the Philosophical Magazine for January 1865, I proved the proposition that "all waves in which molecular rotation is null begin to break when the two slopes of the crest meet at right angles."

2. I have now to state the purely geometrical proposition of which that mechanical proposition is a consequence. *If a plane curve which fulfils the condition* $\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}=0$ *cuts itself in a double point, it does so at right angles.*

3. The following is the demonstration. It is well known that the inclination of any plane curve to the axes at an ordinary point is given by the equation

$$\frac{d\phi}{dx}dx+\frac{d\phi}{dy}dy=0;$$

also that at a double point $\frac{d\phi}{dx}$ and $\frac{d\phi}{dy}$ both vanish, so that the inclinations of the two branches to the axes are given by the two roots of the quadratic equation

$$\frac{d^2\phi}{dx^2}\cdot dx^2+2\frac{d^2\phi}{dx\,dy}\cdot dx\,dy+\frac{d^2\phi}{dy^2}\cdot dy^2=0;$$

whence it follows that the product of the two values of $\frac{dy}{dx}$, which are the two values of the tangent of the inclination to the axis of $x$, is $=\frac{\frac{d^2\phi}{dx^2}}{\frac{d^2\phi}{dy^2}}$. In a curve which fulfils the before-mentioned condition, the value of that product is $-1$; and when such is the case with the product of the tangents of two angles, the difference of those angles is a right angle; therefore the two branches cut each other at right angles. Q.E.D.

4. The proposition just demonstrated is so simple and so obvious, that

pials, corresponds homologically with that which, as a general rule, is most persistent in the typical diphyodonts, including Man, viz. the posterior milk-molar, replaced by the posterior permanent premolar.